ON COMPLEMENTED SUBSPACES OF m

BY

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ABSTRACT

It is proved that an infinite dimensional subspace of m is complemented in m if and only if it is isomorphic to m.

Let Y be a Banach space and let X be a closed linear subspace of Y. We say that X is complemented in Y if there is a bounded linear projection from Y onto X. A Banach space is called a $\mathfrak{P}$ space if it is complemented in every Banach space containing it. For a set Γ let $m(\Gamma)$ denote the space of bounded scalar-valued functions on Γ with the supremum norm. Since every Banach space is isometric to a subspace of $m(\Gamma)$ for a suitable Γ and since every $m(\Gamma)$ is a $\mathfrak{P}$ space it is easily seen and well known (cf. [3, p. 94]) that a Banach space X is a $\mathfrak{P}$ space if and only if X is isomorphic to a complemented subspace of some $m(\Gamma)$. The question of functional representation of $\mathfrak{P}$ spaces has been considered by many authors but is still open. The main known results in this direction are contained in [5] and its references. The purpose of the present note is to settle the question of the structure of the complemented subspaces of $m(\Gamma)$ if Γ is countably infinite (for this Γ we denote, as usual, $m(\Gamma)$ by m).

THEOREM. *An infinite dimensional subspace X of m is complemented in m if and only if X is isomorphic to m.*

The *if* part of the theorem is trivial since m is a $\mathfrak{P}$ space. The theorem answers a question of Pełczyński [5] (cf. also [2]). Pełczyński proved in [5] a result similar to the *only if* part of the theorem for l_p, $1 \leqq p < \infty$, and c_0. We shall use in the proof of our theorem the following two results of Pełczyński.

LEMMA 1. *Let X be an infinite dimensional $\mathfrak{P}$ space. Then X contains a subspace isomorphic to c_0*

For a proof see [5, p. 222] or [6].

LEMMA 2. *Let X be a complemented subspace of m and assume that X has a subspace isomorphic to m. Then X is isomorphic to m.*

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For a proof see [5, p. 222].

We do not use essentially new methods here. Lemma 5 below is closely related to the paper [1] of Bessaga and Pełczyński. In the proof of the theorem itself we use an idea of Nakamura and Kakutani [4] (cf. also [7] and [8]). Our approach can be used to get some information on general $\mathfrak{P}$ spaces. We do not, however, treat general $\mathfrak{P}$ spaces here since it seems that the solution of the problem of characterizing general $\mathfrak{P}$ spaces will need other methods. Our reason for believing this is the observation in [5, p. 223] that there are $\mathfrak{P}$ spaces which are not isomorphic to $m(\Gamma)$ for any Γ.

By the w^* topology of m we understand the topology induced on m by the elements of l_1. For $x \in m$ and an integer i, $x(i)$ denotes the i-th coordinate of x. For $x \in m$ and $\varepsilon > 0$ we put $N(x,\varepsilon) = \{i; |x(i)| > \varepsilon\}$ and $M(x,\varepsilon) = \{i; |x(i)| \leqq \varepsilon\}$. The scalar field (real or complex) is denoted by R.

Lemma 3. *Let $\{x_k\}_{k=1}^{\infty}$ be a sequence of elements in m such that for some constant K*

$$\text{(1)} \qquad \Big\| \sum_{k=1}^{n} \lambda_k x_k \Big\| \leqq K \sup_k |\lambda_k| \quad \textit{for all } \{\lambda_k\}_{k=1}^{n} \subset R, \qquad n = 1, 2, \cdots.$$

Then for every bounded sequence $\{\lambda_k\}_{k=1}^{\infty} \subset R$ the series $\sum_{k=1}^{\infty} \lambda_k x_k$ converges in the w^* topology to an element of norm $\leqq K \sup_k |\lambda_k|$ in m.

Proof. Obvious.

Lemma 4. *Let $\{x_k\}_{k=1}^{\infty}$ be a sequence of elements in m such that* (1) *holds and such that for some $H > 0$*

$$\text{(2)} \qquad \|x_k\| \geqq H \qquad k = 1, 2, \cdots.$$

Then for every $\varepsilon > 0$ and $\theta < H$ there is an index k such that

$$\text{(3)} \qquad A_k = \{h; M(x_k, \varepsilon) \cap N(x_h, \theta) \neq \emptyset\}$$

is an infinite set.

Proof. By (1) we get that for every i, $\sum_k |x_k(i)| \leqq K$ and hence the intersection of $r > K/\varepsilon$ of sets of the form $N(x_k, \varepsilon)$ is empty. Assume that A_k, $k = 1, \cdots, r$, are finite sets. Then $A = \bigcup_{k=1}^{r} A_k$ is a finite set. For $h \notin A$, $N(x_h, \theta) \subset \bigcap_{k=1}^{r} N(x_k, \varepsilon) = \emptyset$ but this contradicts (2).

Lemma 5. *Let $\{x_k\}_{k=1}^{\infty}$ be a sequence of elements in m such that* (1) *holds and $\|x_k\| > 2$ for every k. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_k\}$ such that*

$$\text{(4)} \qquad \sup_k |\lambda_k| \leqq \Big\| \sum_{k=1}^{\infty} \lambda_k x_{n_k} \Big\| \leqq K \sup_k |\lambda_k|,$$

for every bounded sequence $\{\lambda_k\}_{k=1}^{\infty}$ *of scalars.*

Proof. The right inequality of (4) is valid for every subsequence $\{x_{n_k}\}$ by Lemma 3, so we have to consider only the left inequality of (4).

By Lemma 4 there is an n_1 so that $B_1 = \{h;\ M(x_{n_1}, 1/4) \cap N(x_h, 7/4) \neq \emptyset\}$ is infinite. Next we choose an integer i_1 such that $|x_{n_1}(i_1)| \geqq 2$. Since $\sum_{k=1}^{\infty} |x_k(i_1)| \leqq K$ there is an infinite subset C_1 of B_1 such that $\sum_{k \in C_1} |x_k(i_1)| < 1/4$. The set $M_1 = M(x_{n_1}, 1/4)$ is an infinite set. Indeed, otherwise we would have an infinite number of $h \in B_1$ such that $|x_h(i)| \geqq 7/4$ for some fixed $i \in M_1$ and this contradicts the fact that $\sum_h |x_h(i)| \leqq K$. Let $x_{k,1}$ be the restriction of x_k to M_1, $k = 1, 2, \cdots$ (i.e. $x_{k,1}(i)$ is defined only for $i \in M_1$ and for these i, $x_{k,1}(i) = x_k(i)$). By the definition of M_1, $\|x_{k,1}\| > 7/4$ for $k \in B_1$ and hence for $k \in C_1$. By applying Lemma 4 to $\{x_{k,1}\}_{k \in C_1}$ we get an $n_2 \in C_1$ such that

$$B_2 = C_1 \cap \{h; M_1 \cap M(x_{n_2}, 1/4^2) \cap N(x_h, 7/4 - 1/4^2) \neq \emptyset\}$$

is an infinite set. Let $i_2 \in M_1$ be such that $|x_{n_2}(i_2)| \geqq 7/4$ and let C_2 be an infinite subset of B_2 such that $\sum_{k \in C_2} |x_k(i_2)| < 1/4^2$. Set $M_2 = M_1 \cap M(x_{n_2}, 1/4^2)$ and $x_{k\,2}$ the restriction of x_k to M_2, $k = 1, 2, \cdots$. Continuing inductively we get a subsequence $\{x_{n_k}\}$ of $\{x_k\}$ and a sequence of integers i_k such that

$$|x_{n_k}(i_k)| \geqq 2 - 1/4 - \cdots - 1/4^{k-1} > 5/3 \tag{5}$$

$$\sum_{k=j+1}^{\infty} |x_{n_k}(i_j)| \leqq 1/4^j \tag{6}$$

$$|x_{n_k}(i_j)| \leqq 1/4^k \qquad \text{for } j > k \tag{7}$$

By (6) and (7) $\sum_{k \neq j} |x_{n_k}(i_j)| < 1/3$. Let now $\{\lambda_k\}_{k=1}^{\infty}$ be any sequence of scalars such that $\sup_k |\lambda_k| = 1$ and let j be such that $|\lambda_j| > 4/5$. Then

$$\begin{aligned}\Big\|\sum_{k=1}^{\infty} \lambda_k x_{n_k}\Big\| &\geqq \Big|\sum_{k=1}^{\infty} \lambda_k x_{n_k}(i_j)\Big| \geqq |\lambda_j|\,|x_{n_j}(i_j)| - \sum_{k \neq j} |\lambda_k|\,|x_{n_k}(i_j)| \\ &\geqq 5|\lambda_j|/3 - \sum_{k \neq j} |x_{n_k}(i_j)| \geqq 1,\end{aligned}$$

and this concludes the proof of the lemma.

Proof of the theorem. Let X be an infinite dimensional complemented subspace of m. By Lemma 1 X contains a subspace isomorphic to c_0. By Lemma 5 it follows that there is a sequence $\{x_k\}_{k=1}^{\infty}$ in X and a constant K such that $\sup_k |\lambda_k| \leqq$ $\leqq \|\sum_{k=1}^{\infty} \lambda_k x_k\| \leqq K \sup_k |\lambda_k|$ for every bounded sequence of scalars $\{\lambda_k\}_{k=1}^{\infty}$. (The series $\sum \lambda_k x_k$ converges in the w^* topology).

Let $\{N_\gamma\}_{\gamma \in \Gamma}$ be an uncountable collection of infinite subsets of the integers such that $N_\beta \cap N_\gamma$ is finite whenever $\beta \neq \gamma$. The existence of such a set is well known (cf. [7], [8] and the references there). For each $\gamma \in \Gamma$ let X_γ be the subspace of m consisting of the elements of the form $\sum_{k \in N_\gamma} \lambda_k x_k$ where $\{\lambda_k\}_{k \in N_\gamma}$ is a bounded set of scalars. Clearly every X_γ is isomorphic to m. We shall prove that $X_\gamma \subset X$ for some γ and this will conclude the proof of the theorem (by Lemma 2).

Let $\phi: m \to m/X$ be the quotient map. Since X is complemented in m, m/X is isomorphic to a subspace of m. Hence there is a countable set of functionals $\{f_j\}_{j=1}^\infty$ in $(m/X)^*$ such that $f_j(u) = 0$ for every j ($u \in m/X$) implies $u = 0$. Assume that for every $\gamma \in \Gamma$ there is an $x_\gamma = \sum_{k \in N_\gamma} \lambda_k^\gamma x_k$ such that $\| x_\gamma \| = 1$ and $x_\gamma \notin X$ i.e. $\phi(x_\gamma) \neq 0$. We claim that for every choice of signs ε_i and every finite set $\{x_{\gamma_i}\}_{i=1}^n$

$$\left\| \sum_{i=1}^{n} \varepsilon_i \phi(x_{\gamma_i}) \right\| \leqq K. \tag{8}$$

Indeed, by our assumption on the N_γ there is a finite set M_0 such that $N_{\gamma_i} \subset M_0 \cup M_i$, $i = 1, \cdots, n$, and $M_i \cap M_k = \emptyset$ for $i \neq k$. Put, for $i = 1, \cdots, n$, $x_{\gamma_i} = y_i + z_i$ where $y_i = \sum_{k \in M_0} \lambda_k^{\gamma_i} x_k$ and $z_i = \sum_{k \in M_i} \lambda_k^{\gamma_i} x_k$. Every y_i belongs to X and hence $\phi(x_{\gamma_i}) = \phi(z_i)$. Since $\| \sum_{i=1}^n \varepsilon_i z_i \| \leqq K$ for every choice of signs we get (8).

From (8) it follows that for every $f \in (m/X)^*$, $\sum_\gamma |f(\phi(x_\gamma))| \leqq K \|f\|$ and in particular there is only a countable set of $\gamma \in \Gamma$ such that $f(\phi(x_\gamma)) \neq 0$. It follows that there is only a countable number of $\gamma \in \Gamma$ such that $f_j(\phi(x_\gamma)) \neq 0$ for some j and this contradicts our choice of the f_j and the assumption that $\phi(x_\gamma) \neq 0$ for every γ.

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